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Nonrelativistic Wave Equations With Gauge Fields

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We illustrate a metric formulation of Galilean invariance by constructing wave equations with gauge fields. It consists of expressing nonrelativistic equations in a covariant form, but with a five-dimensional Riemannian manifold. First we use the tensorial expressions of electromagnetism to obtain the two Galilean limits of electromagnetism found previously by Le Bellac and Lévy-Leblond. Then we examine the nonrelativistic version of the linear Dirac wave equation. With an Abelian gauge field we find, in a weak field approximation, the Pauli equation as well as the spin–orbit interaction and a part reminiscent of the Darwin term. We also propose a generalized model involving the interaction of the Dirac field with a non-Abelian gauge field; the SU(2) Hamiltonian is given as an example.

KEY WORDS: Galilean invariance; Riemannian geometry; gauge theory; wave equations.

1. INTRODUCTION

Almost a century ago, Galilean relativity was superseded in a spectacular fashion by Einstein's theory for the description of phenomena involving velocities close to the speed of light. However, there exists a wealth of low-energy systems, particularly in condensed matter physics and low-energy nuclear physics, where Galilean invariance cannot simply be ignored. Thus, any new method or result involving Galilean invariance is likely to be useful. Considering the fact that Galilean relativity has been known for nearly 200 years prior to Einstein's relativity, it may appear surprising that the group theory underlying relativistic theories, the Poincaré group, has been thoroughly investigated (Bargmann, 1947; Gelfand *et al.*, 1963; Wigner, 1939) long before its nonrelativistic counterpart, the Galilei group (Inönü and Wigner, 1952; Lévy-Leblond, 1963, 1971). Although

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the *principle* of relativity has been recognized first in the low velocity regime, the mathematical developments for the Poincaré group have always preceded those for the Galilean group. Likewise, the present article is part of a program, which borrows the tensor calculus typically utilized within Lorentz-covariant models, in order to construct Galilei-invariant theories. Our purpose was to retrieve nonrelativistic models by starting with a manifest Galilei-covariant theory, although in a five dimensional Riemannian manifold. Such a unified formalism, where the physical theories look as similar as possible in their Lorentzian and Galilean versions, would show more manifestly how some concepts or techniques can be shared between the two theories. Thus it is not surprising that four-dimensional covariant descriptions of Newtonian mechanics and gravitation were given soon after Einstein's formulation of special relativity (see, for instance, Carton, 1923 (1924) and the review in Havas (1964)). This approach was used to explain the appearance of Poincaré symmetries in the description of *nonrelativistic* membranes by $2 + 1$ dimensional field theory or, equivalently, irrotational isentropic fluid motion for a specific potential (Bazeia and Jackiw, 1998; Bordemann and Hoppe, 1993; Hassaïne and Harváthy, 2000, 2001; Jackiw, 2002; Jackiw and Polychronakos, 1999).

In this paper we use a similar approach to consider nonrelativistic wave equations for gauge fields and their interaction with fermion fields. We are more oriented toward physical applications than many earlier papers that dealt with this five-dimensional approach. Most results obtained here are already in the literature; our purpose here is to express them in a Galilei-covariant form. In Section 2, we use the manifest covariant tensorial expressions of Maxwell electrodynamics and obtain thereby the two nonrelativistic versions of electromagnetism derived nearly three decades ago by Le Bellac and Lévy-Leblond (1973). In Section 3, we turn to first-order field equations: the Dirac equation for a fermionic field coupled to an external (Abelian) electromagnetic field, and for a fermion coupled to a non-Abelian gauge field. We do not investigate Galilean theories of gravitation, which are already investigated (Duval *et al.*, 1985 Julia and Nicolai, 1995; Nurowski *et al.*, 1999). Although reminiscent of the old attempts to geometrize all fields into a unified field theory à la Kaluza–Klein, here the aim of the fifth dimension is completely different, namely, to combine the relativistic and the Galilean structures. To our knowledge, the use of five dimensions in this context was originally mentioned by Lévy-Leblond (1971), Pinski (1968), and Soper (1976) and investigated more thoroughly by Künzle and Duval (1994) and Pinski (1968). An interesting interpretation of the fifth parameter is in the first of Kapuścik (1986). However hereafter we shall follow a formulation introduced by Takahashi and his collaborators (de Montigny *et al.*, 2000, 2001a,b; Omote *et al.*, 1989; Santana *et al.*, 1998; Takahashi, 1987, 1988a,b), based on a five-dimensional space such that a Galilean boost with relative velocity $\mathbf{V} = (V_1, V_2, V_3)$ acts on a *Galilei-vector* $(\mathbf{x}, t, s)$ as

$$
\begin{aligned}
\mathbf{x}' &= \mathbf{x} - \mathbf{V}t, \\
t' &= t, \\
s' &= s - \mathbf{V} \cdot \mathbf{x} + \frac{1}{2}\mathbf{V}^2 t.
\end{aligned} \tag{1}
$$

The scalar product,

$$
(A|B) = A^{\mu} B_{\mu} \equiv \mathbf{A} \cdot \mathbf{B} - A_4 B_5 - A_5 B_4, \tag{2}
$$

of two Galilei-vectors A and B is invariant under the transformation, Eq. (1). This suggests a method to base the tensor calculus on the metric

$$
g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3}
$$

Hereafter we refer to this as the *Galilean metric*. It may be artificial to refer to a "Galilean tensor calculus" since the usual tensor operations and index techniques simply cannot be performed in the Newtonian space–time, but only in five dimensions. Also, the Newtonian space–time must be embedded into that space according to Eq. (1). As mentioned by Omote *et al.* (1989), this metric may be diagonalized into diag$(+, +, +, +, -)$ so that the Galilean covariance is achieved by embedding the ordinary Newtonian space into a $4 + 1$ Minkowski space. Susskind (1968) has noticed that working with light-cone coordinates in a $(d + 1, 1)$ Minkowski space–time reduces to Galilean invariance in $(d, 1)$ dimensions. This corroborates the unified formalism of Künzle and Duval ((1994) and references therein) where both the Lorentzian and the Newtonian space–time can be described in terms of a five-dimensional Lorentz metric with its Levi-Civita connection together with a covariantly constant vector field that is null in the Galilei case, and spacelike in the Lorentz case. The four-dimensional space–time then arises as the quotient manifold of the orbits of the corresponding vector field. The five-dimensional Lorentzian metric is (see the end of Omote *et al.* (1989).

$$
(g_{\mathrm{Lor}})_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1/c^2 \end{pmatrix}, \tag{4}
$$

$$(g_{\mathrm{Lor}})^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/c^2 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{5}$$

and leads to the metric, Eq. (3), when c approaches infinity.

The transformation in Eq. (1) can be written in matrix form for the components of any five-vector as

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}, \tag{6}$$

where μ denotes the row and ν the column (so that Λ^{μ}_{ν} is the $(\mu\nu)$-entry) or

$$\begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \\ x'^4 \\ x'^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -V_1 & 0 \\ 0 & 1 & 0 & -V_2 & 0 \\ 0 & 0 & 0 & -V_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -V_1 & -V_2 & -V_3 & \frac{1}{2}\mathbf{V}^2 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix}. \tag{7}$$

The same transformation can be written in matrix form for any five-oneform as

$$x'_{\mu} = \Lambda^{\nu}_{\mu} x_{\nu}, \tag{8}$$

where μ now denotes the column and ν the row (that is Λ^{ν}_{μ} is the $(\nu\mu)$-entry), or

$$(x'_1, x'_2, x'_3, x'_4, x'_5) = (x_1, x_2, x_3, x_4, x_5) \begin{pmatrix} 1 & 0 & 0 & V_1 & 0 \\ 0 & 1 & 0 & V_2 & 0 \\ 0 & 0 & 1 & V_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ V_1 & V_2 & V_3 & \frac{1}{2}\mathbf{V}^2 & 1 \end{pmatrix}. \tag{9}$$

These matrix elements are calculated by using $x'_{\mu} = g_{\mu\alpha} x'^{\alpha} = \overbrace{g_{\mu\alpha} \Lambda^{\alpha}_{\beta} g^{\beta\nu}}^{\Lambda^{\nu}_{\mu}} x_{\nu}$.

Note that the units of the additional coordinate s are [length^2]/[time]. Sometimes it is useful to define a five-vector $(x^1, \ldots, x^5)$, where each component has the dimension of length, from Eq. (1) as

$$(x^1, \ldots, x^5) = \left(\mathbf{x}, v_4 t, \frac{s}{v_5} \right), \tag{10}$$

where both v_4 and v_5 have units of a velocity. The units of the *covariant components* of the five-vector $(x^1, \ldots, x^5) = (\mathbf{x}, t, s)$ are

$$[\mathbf{x}] = L, \quad [x^4] = T, \quad [x^5] = \frac{L^2}{T}, \tag{11}$$

(where L and T represent units of length and time, respectively) whereas Eq. (9) shows that the units of the *contravariant components* are

$$[\mathbf{x}] = L, \quad [x_4] = \frac{L^2}{T}, \quad [x_5] = T. \tag{12}$$

Whether we consider a five-vector or a five-oneform these relative units between each components must be kept in mind, as illustrated in the next section.

In most of this paper (except in Section 2) we shall use the embedding

$$(\mathbf{x}, t) \to x^\mu = (\mathbf{x}, t, s). \tag{13}$$

Other developments are discussed in de Montigny *et al.* (2001a) and Santand *et al.* (1998). Using the following definition for the five-momentum:

$$p_\mu \equiv -i\partial_\mu = (-i\nabla, -i\partial_t, -i\partial_s), \tag{14}$$

and with the usual identification $E = i\partial_t$, and writing $m = i\partial_s$, we obtain

$$\begin{aligned} p_\mu &= (\mathbf{p}, -E, -m), \\ p^\mu &= g^{\mu\nu} p_\nu = (\mathbf{p}, m, E). \end{aligned} \tag{15}$$

We use the convention $\hbar = 1$ throughout the whole paper. Also we use the embedding (15) in most of this paper, i.e., $p_4 = -E$ and $p_5 = -m$. Thereupon the mass does not enter as an external parameter, but as a remnant of the fifth component of the particle's momentum, although we started from an apparently massless theory in five dimensions.

2. GALILEAN ELECTROMAGNETISM

The purpose of this section is to illustrate the elegance of the five-dimensional approach for a gauge field without any interaction with a fermionic field. Specifically we retrieve the two Galilean limits of electromagnetism obtained by Le Bellac and Lévy-Leblond (1973). Their purpose was to write down the laws of electromagnetism by making use of Galilei relativity instead of Einstein's relativity, the latter leading to the electromagnetic theory as we know it today. As they put it, the laws obtained thereby could have been formulated by a physicist in the mid-nineteenth century. Here we retrieve their results by using the tensorial form of Maxwell equations.

As stated in Le Bellac and Lévy-Leblond (1973), the Lorentz transformation of a four-vector $(u^0, \mathbf{u})$ (Goldstein, 1980, Chap. 7)

$$u'^0 = \gamma\left(u^0 - \frac{1}{c}\mathbf{V}\cdot\mathbf{u}\right),$$

$$\mathbf{u}' = \mathbf{u} - \gamma\frac{\mathbf{V}}{c}u^0 + \frac{\mathbf{V}}{\mathbf{V}^2}(\gamma - 1)\mathbf{V}\cdot\mathbf{u}, \tag{16}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - \mathbf{V}^2/c^2}}, \tag{17}$$

with an arbitrary relative velocity $\mathbf{V}$, admits two well-defined Galilean limits. The speed of light in the vacuum is denoted c. One limit is for timelike vectors

$$u'^0 = u^0,$$

$$\mathbf{u}' = \mathbf{u} - \frac{1}{c}\mathbf{V}u^0, \tag{18}$$

and we shall see that it corresponds to the so-called *electric* limit. The second limit is for spacelike vectors

$$u'^0 = u^0 - \frac{1}{c}\mathbf{V}\cdot\mathbf{u},$$

$$\mathbf{u}' = \mathbf{u}, \tag{19}$$

and will be associated to the *magnetic* limit. Although the space–time coordinates can be described by timelike vectors only, other vectors, such as the four-potential and four-current, are compatible with the two limits. The existence of two Galilean limits of electromagnetism is but a warning that there is more to nonrelativistic theories than just taking the speed of light approaching infinity. Another example is that if one neglects to enforce the condition that a nonrelativistic limit involves not only low-velocity phenomena but also large timelike intervals then one obtains different kinematics, referred to as Carroll kinematics (Lévy-Lebland, 1965). The existence of events physically connected by large spacelike intervals would imply loss of causality, among other things. Other such kinematics, all obtained as some limit of the de Sitter Lie algebra, have been classified in Bacry and Lévy-Leblond (1968).

Now let us set up the five-dimensional quantities that allow us to retrieve the two Galilean limits of electromagnetism. The Galilean tensor calculus is based on the fact that a five-vector (i.e., with upper indices) transforms as in Eq. (1) with the underlying metric (3). Throughout this section, we use a rather trivial embedding of the Newtonian space–time into this five-dimensional space.

$$(\mathbf{x}, t) \hookrightarrow x = (\mathbf{x}, t, 0), \tag{20}$$

so that

$$\partial_k = \nabla_k, \quad \partial_4 = \partial_t, \quad \partial_5 = 0. \tag{21}$$

We obtain the two Galilean limits by defining two embeddings of the five-potential

$$A_\mu = (\mathbf{A}, A_4, A_5). \tag{22}$$

Under the transformation in Eq. (1) its components transform, from Eq. (9), as

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \mathbf{V} A_5, \\ A_{4'} &= A_4 + \mathbf{V} \cdot \mathbf{A} + \frac{1}{2}\mathbf{V}^2 A_5, \\ A_{5'} &= A_5. \end{aligned} \tag{23}$$

The potential defines the five-dimensional electromagnetic antisymmetric tensor,

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{24}$$

which can be written as

$$F_{\mu\nu} = \begin{pmatrix} 0 & b_3 & -b_2 & c_1 & d_1 \\ -b_3 & 0 & b_1 & c_2 & d_2 \\ b_2 & -b_1 & 0 & c_3 & d_3 \\ -c_1 & -c_2 & -c_3 & 0 & a \\ -d_1 & -d_2 & -d_3 & -a & 0 \end{pmatrix}. \tag{25}$$

From Eq. (24) we have

$$\begin{aligned} \mathbf{b} &= \nabla \times \mathbf{A}, \\ \mathbf{c} &= \nabla A_4 - \partial_4 \mathbf{A}, \\ \mathbf{d} &= \nabla A_5 - \partial_5 \mathbf{A}, \\ a &= \partial_4 A_5 - \partial_5 A_4. \end{aligned} \tag{26}$$

Anticipating that the components **b** correspond to the magnetic field **B**, the units of the different components are

$$\begin{aligned} [\mathbf{b}] &= \frac{M}{QT} = [a], \\ [\mathbf{c}] &= \frac{ML}{T^2 Q}, \\ [\mathbf{d}] &= \frac{M}{QL}, \end{aligned} \tag{27}$$

where Q and M denote units of charge and mass, respectively.

The five-current

$$j_\mu = (\mathbf{j},\, j_4,\, j_5), \tag{28}$$

transforms under the transformation, Eq. (1), as

$$\begin{aligned} \mathbf{j}' &= \mathbf{j} + \mathbf{V} j_5,\\ j_{4'} &= j_4 + \mathbf{V}\cdot\mathbf{j} + \frac{1}{2}\mathbf{V}^2 j_5,\\ j_{5'} &= j_5. \end{aligned} \tag{29}$$

The continuity equation takes the form

$$\partial^\mu j_\mu = \nabla\cdot\mathbf{j} - \partial_4 j_5 - \partial_5 j_4 = 0. \tag{30}$$

In the presence of sources, the Maxwell equations are

$$\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} = 0, \tag{31}$$

and

$$\partial_\nu F^{\mu\nu} = j^\mu, \tag{32}$$

so that in terms of the components of F defined in Eq. (25), we find, from Eq. (31)

$$\begin{aligned} &\nabla\cdot\mathbf{b} = 0,\\ &\nabla\times\mathbf{c} + \partial_4\mathbf{b} = \mathbf{0},\\ &\nabla\times\mathbf{d} + \partial_5\mathbf{b} = \mathbf{0},\\ &\nabla a - \partial_4\mathbf{d} + \partial_5\mathbf{c} = \mathbf{0}, \end{aligned} \tag{33}$$

whereas Eq. (32) reduces to

$$\begin{aligned} &\nabla\times\mathbf{b} - \partial_5\mathbf{c} - \partial_4\mathbf{d} = \mathbf{j},\\ &\nabla\cdot\mathbf{c} - \partial_4 a = -j_4,\\ &\nabla\cdot\mathbf{d} - \partial_5 a = -j_5, \end{aligned} \tag{34}$$

From $F_{\mu'\nu'} = \Lambda^\alpha_{\mu'}\Lambda^\beta_{\nu'}F_{\alpha\beta}$ the entries of F in Eq. (9) transform as

$$\begin{aligned} a' &= a + \mathbf{V}\cdot\mathbf{d},\\ \mathbf{b}' &= \mathbf{b} - \mathbf{V}\times\mathbf{d},\\ \mathbf{c}' &= \mathbf{c} + \mathbf{V}\times\mathbf{b} + \frac{1}{2}\mathbf{V}^2\mathbf{d} - a\mathbf{V} - \mathbf{V}(\mathbf{V}\cdot\mathbf{d}),\\ \mathbf{d}' &= \mathbf{d}. \end{aligned} \tag{35}$$

Finally let us discuss how the Lorentz force **f** is contained within a five-force proportional to the velocity and the electromagnetic field, that is

$$f^{\mu} \propto q F^{\mu}_{\nu} v^{\nu}, \tag{36}$$

where q denotes the charge of the particle moving in the field. The spatial components are

$$f^{j} \propto q(F^{jk} v^{k} - F^{j5} v^{4} - F^{j4} v^{5}). \tag{37}$$

Using Eq. (25) it gives

$$\mathbf{f} \propto q(\mathbf{v} \times \mathbf{b} - \mathbf{d} v^{4} - \mathbf{c} v^{5}). \tag{38}$$

Now we shall define two embeddings of the usual four-potential into the five dimensional version, Eq. (22), and consider their effect on the equations listed above. This will result in a geometrical formulation of the 'electric' and 'magnetic' limits obtained in Le Bellac and Lévy-Leblond (1973).

2.1. Electric Limit

In Newtonian space–time, the electric limit is characterized by four-potential and four-current vectors which are timelike, that is, their time component is much larger than the length of their spatial components. In the setting described above it corresponds to defining the embedding of the potentials and currents as

$$(\mathbf{A}_e, \phi_e) \hookrightarrow A_e = \left(\mathbf{A}_e, 0, -\frac{1}{k_1}\phi_e\right), \tag{39}$$

and

$$(\mathbf{j}_e, \rho_e) \hookrightarrow je = (k_2 \mathbf{j}_e, 0, -k_2 \rho_e), \tag{40}$$

respectively. The constants k_1 and k_2 have dimensions $\frac{L^2}{T^2}$ and $\frac{ML}{Q^2}$, respectively. Note that the units of k_2 are the same as the constant of permittivity μ_0.

From Eqs. (23) and (39) we find that

$$\mathbf{A}'_e = \mathbf{A}_e - \frac{1}{k_1}\mathbf{V}\phi_e \quad \text{and} \quad \phi'_e = \phi_e, \tag{41}$$

in agreement with Le Bellac and Lévy-Leblond (1973) as long as we choose $k_1 \equiv \frac{1}{\mu_0 \epsilon_0}$. Similarly we find from Eqs. (29) and (40) that

$$\mathbf{j}'_e = \mathbf{j}_e - \mathbf{V}\rho_e \quad \text{and} \quad \rho'_e = \rho_e, \tag{42}$$

as in Le Bellac and Lévy-Leblond (1973). The continuity Eq. (30) then reads

$$\nabla \cdot \mathbf{j} - \partial_4 j_5 - \partial_5 j_4 = \nabla \cdot \mathbf{j}_e + \partial_t \rho_e = 0, \tag{43}$$

where Eq. (20) has been used.

Next we must define the electric and magnetic fields. It is clear from the first line of Eq. (26) that we have

$$\mathbf{B}_e \equiv \mathbf{b} = \nabla \times \mathbf{A}_e. \tag{44}$$

The electric field is defined as the component $\mathbf{d}$, so that from the third line of Eq. (26) we have

$$\mathbf{E}_e \equiv k_1 \mathbf{d} = \frac{1}{\mu_0 \epsilon_0} \mathbf{d} = -\nabla \phi_e. \tag{45}$$

From Eq. (26) we note that $\mathbf{c} = -\partial_t \mathbf{A}_e$ and $a = -\frac{1}{\mu_0 \epsilon_0} \partial_t \phi_e$. Then Eq. (35) shows that

$$\begin{aligned} \mathbf{E}'_e &= \mathbf{E}_e, \\ \mathbf{B}'_e &= \mathbf{B}_e - \mu_0 \epsilon_0 \mathbf{V} \times \mathbf{E}_e \end{aligned} \tag{46}$$

as in Le Bellac and Lévy-Leblond (1973). The Maxwell equations are obtained from Eqs. (33) and (34) by defining $k_2 \equiv \mu_0$. The new equations thus obtained are

$$\begin{aligned} \nabla \times \mathbf{E}_e &= \mathbf{0}, \\ \nabla \cdot \mathbf{B}_e &= 0, \\ \nabla \times \mathbf{B}_e - \mu_0 \epsilon_0 \partial_t \mathbf{E}_e &= \mu_0 \mathbf{j}_e, \\ \nabla \cdot \mathbf{E}_e &= \frac{1}{\epsilon_0} \rho_e. \end{aligned} \tag{47}$$

This is Eq. (2.8) Le Bellac and Lévy-Leblond (1973). Note that the second line of Eq. (34) provides a condition similar to Lorentz gauge fixing, $\nabla \cdot \mathbf{A}_e = \mu_0 \epsilon_0 \partial_t \phi_e$.

The Lorentz force in this limit is obtained by using $v^4 = v$, $v^5 = 0$ (because of Eq. (20)) and $\mathbf{d} = \mu_0 \epsilon_0 \mathbf{E}_e$ so that Eq. (38) becomes

$$\mathbf{f}_e \propto q\mathbf{E}_e + q\mathbf{v} \times \mathbf{B}_e \approx q\mathbf{E}_e, \tag{48}$$

in the small $\mathbf{B}_e$ limit, as in Le Bellac and Lévy-Leblond (1973).

2.2. Magnetic Limit

This nonrelativistic limit is characterized by spacelike four-potential and four-current vectors; their time component is small compared to the length of their spatial components. Hereafter we show that it corresponds to defining the embedding of the potentials and currents as

$$(\mathbf{A}_m, \phi_m) \hookrightarrow A_m = (\mathbf{A}_m, -\phi_m, 0), \tag{49}$$

and

$$(\mathbf{j}_m, \rho_m) \hookrightarrow j_m = (k_3 \mathbf{j}_m, -k_4 \rho_m, 0), \tag{50}$$

respectively. Since the four- and five-components have different units, the constants included above are different from those in Eqs. (39) and (40). The constants k_3 and k_4 now have units $\frac{ML}{Q^2}$ and $\frac{ML}{Q^2}\frac{L^2}{T^2}$, respectively. Now the units of k_3 are the same as the constant of permittivity μ_0, whereas those of k_4 are like the inverse of ϵ_0. Also note the absence of such constants in equation (49), since the units are already compatible. As before, we find that equations of Le Bellac and Lévy-Leblond (1973) are obtained by defining $k_3 \equiv \mu_0$ and $k_4 \equiv 1/\epsilon_0$.

From Eqs. (23) and (49) we find that

$$\mathbf{A}'_m = \mathbf{A}_m \quad \text{and} \quad \phi'_m = \phi_m - \mathbf{V} \cdot \mathbf{A}_m. \tag{51}$$

Similarly Eqs. (29) and (50) lead to

$$\mathbf{j}'_m = \mathbf{j}_m \quad \text{and} \quad \rho'_m = \rho_m - \mu_0\epsilon_0\mathbf{V} \cdot \mathbf{j}_m. \tag{52}$$

Using the continuity Eq. (30) and the embedding, Eq. (21), we find

$$\nabla \cdot \mathbf{j} - \partial_4 j_5 - \partial_5 j_4 = \nabla \cdot \mathbf{j}_m = 0, \tag{53}$$

which is Eq. (2.16) of Le Bellac and Lévy-Leblond (1973) and shows that the current $\mathbf{j}_m$ cannot be related to a transport of charge.

Next we define the electric and magnetic fields. We take

$$\mathbf{B}_m \equiv \mathbf{b} = \nabla \times \mathbf{A}_m, \tag{54}$$

and the electric field is now defined as the component $\mathbf{c}$, so that from the second line of Eq. (26) we obtain

$$\mathbf{E}_m \equiv \mathbf{c} = -\nabla\phi_m - \partial_t\mathbf{A}_m. \tag{55}$$

From Eq. (26) we note that $\mathbf{d} = 0$ and $a = 0$. Then Eq. (35) shows that

$$\begin{aligned} \mathbf{E}'_m &= \mathbf{E}_m + \mathbf{V} \times \mathbf{B}_m, \\ \mathbf{B}'_m &= \mathbf{B}_m \end{aligned} \tag{56}$$

as in Le Bellac and Lévy-Leblond (1973). The Maxwell equations are obtained from Eqs. (33) and (34). The new equations thus obtained are

$$\begin{aligned} \nabla \times \mathbf{E}_m &= -\partial_t\mathbf{B}_m, \\ \nabla \cdot \mathbf{B}_m &= 0, \\ \nabla \times \mathbf{B}_m &= \mu_0\mathbf{j}_m, \\ \nabla \cdot \mathbf{E}_m &= \frac{1}{\epsilon_0}\rho_m, \end{aligned} \tag{57}$$

in agreement with Eq. (2.15) of Le Bellac and Lévy-Leblond (1973).

With $b = \mathbf{B}_m$, $\mathbf{c} = \mathbf{E}_m$ and $\mathbf{d} = 0$, the Lorentz force, Eq. (38) becomes

$$\mathbf{f}_m \propto q\mathbf{v} \times \mathbf{B}_m, \tag{58}$$

as shown in Le Bellac and Lévy-Leblond (1973).

Let us close this section by deriving the nonrelativistic Proca equation, which is a generalization of Maxwell's equations for massive spin one particles. The equation of motion takes a form similar to Eq. (32)

$$\partial^\mu F_{\mu\nu} - m^2 A_\nu = 0, \tag{59}$$

where the second term replaces the current j_ν. Therefore the ensuing equations can be obtained directly from the previous results. In the electric limit we find from Eq. (47) that

$$\begin{aligned} \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= -m^2 \mathbf{A}, \\ \nabla \cdot \mathbf{E} &= -m^2 \phi. \end{aligned} \tag{60}$$

In the magnetic limit, the Proca equation leads to

$$\begin{aligned} \nabla \times \mathbf{B} &= -m^2 \mathbf{A}, \\ \nabla \cdot \mathbf{E} &= -m^2 \phi, \end{aligned} \tag{61}$$

which is similar to Eq. (57) with the current and density replaced by $\mathbf{A}$ and ϕ, respectively.

3. DIRAC EQUATION: SPIN 1/2

In previous papers, we have constructed nonrelativistic Bhabha equations by replacing, in their relativistic form, the Lorentz metric with the Galilean metric (de Montigny *et al.*, 2000, 2001b). This procedure provides nonrelativistic field equations without taking any low-velocity limit but rather by starting with a manifestly covariant equation and by defining an appropriate embedding of the Newtonian space into the Galilei-de Sitter space. Our purpose is to compare the output of our algorithm with well-known low-velocity limits.

A Galilean–Dirac equation has been constructed by Omote *et al.* (1989) essentially by enforcing the anticommutation relations of the gamma matrices with the metric of Eq. (3). Here we first retrieve this equation by starting from the free Hamiltonian and then writing a compatible linear equation as done in Lévy-Leblond (1967).

Let us consider a free particle, with nonrelativistic Hamiltonian

$$E = \frac{\mathbf{p}^2}{2m}. \tag{62}$$

Typically, the transition to quantum mechanics is carried out by replacing the dynamical variables E and $\mathbf{p}$ by the operators

$$E \to i\hbar\frac{\partial}{\partial t},$$
$$\mathbf{p} \to -i\hbar\nabla, \tag{63}$$

so that Eq. (62) leads to the nonrelativistic Schrödinger equation

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t). \tag{64}$$

Hereafter we write a similar, albeit first-order, nonrelativistic wave equation compatible with Eq. (62). Essentially we proceed as in Lévy-Leblond (1967) but we emphasize the use of five dimensions. A similar approach has been used in Kapuścik (1985). We need a linear operator which, when applied twice, is equal to

$$\mathbf{p}^2 - 2mE = 0. \tag{65}$$

Now let us write the corresponding linear wave equation in five dimensions as

$$\not{p}\Psi \equiv (\gamma\cdot\mathbf{p} + \gamma^4 p_4 + \gamma^5 p_5)\Psi(x) = 0, \tag{66}$$

where all the (translation) operators p_μ commute among themselves. In this paper we shall use different versions of Eq. (66). The γ^k are the B^k of Lévy-Leblond (1967), and the matrices γ^4 and γ^5 are related to A and C of Lévy-Leblond (1967) by the diagonalization of the metric (3) to diag $(+ + + + -)$. Note that the minus sign is absent in Lévy-Leblond (1967) because i factors there are included in the operator B_4. Eq. (66) follows from the Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\not{\partial})\Psi. \tag{67}$$

By squaring the operator of Eq. (66) we obtain

$$\left[\frac{\gamma^m\gamma^n + \gamma^n\gamma^m}{2}p_m p_n + (\gamma^m\gamma^4 + \gamma^4\gamma^m)p_m p_4 + (\gamma^m\gamma^5 + \gamma^5\gamma^m)p_m p_5 \right.$$
$$\left. + (\gamma^4\gamma^5 + \gamma^5\gamma^4)p_4 p_5 + (\gamma^4)^2(p_4)^2 + (\gamma^5)^2(p_5)^2\right]\Psi = 0. \tag{68}$$

This can be identified with the corresponding terms of Eq. (65) so that we find

$$\gamma^m\gamma^n + \gamma^n\gamma^m = 2\delta^{mn},$$
$$\gamma^m\gamma^4 + \gamma^4\gamma^m = 0 = \gamma^m\gamma^5 + \gamma^5\gamma^m,$$
$$(\gamma^4\gamma^5 + \gamma^5\gamma^4)p_4 p_5 + (\gamma^4)^2(p_4)^2 + (\gamma^5)^2(p_5)^2 = -2mE. \tag{69}$$

This is satisfied by choosing, for instance,

$$\begin{aligned}
&\{\gamma^4, \gamma^5\} \equiv \gamma^4\gamma^5 + \gamma^5\gamma^4 = -2,\\
&(\gamma^4)^2 = 0 = (\gamma^5)^2,\\
&p_4 p_5 = mE.
\end{aligned} \tag{70}$$

This choice has been used by Omote *et al.* (1989) and we shall use it hereafter. Another choice is

$$(\gamma^4)^2 + (\gamma^5)^2 = -1, \quad \gamma^4\gamma^5 + \gamma^5\gamma^4 = 0, \tag{71}$$

together with the condition

$$(p_4)^2 = (p_5)^2 = mE. \tag{72}$$

We will not consider such alternatives any longer here.

The condition given by Eq. (70) is compatible with the embedding, Eq. (15), so that our nonrelativistic linear wave equation is given by Eq. (66) with the γ matrices such that

$$\begin{aligned}
&\{\gamma^m, \gamma^n\} = 2\delta^{mn},\\
&\{\gamma^m, \gamma^4\} = 0 = \{\gamma^m, \gamma^5\},\\
&\{\gamma^4, \gamma^5\} = -2,\\
&(\gamma^4)^2 = 0 = (\gamma^5)^2,
\end{aligned} \tag{73}$$

or, in a compact form,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{74}$$

with $g^{\mu\nu}$ given by Eq. (3). However it must kept in mind that this holds only if the last row of Eq. (70) is used.

From the general spinor theory, the Clifford algebra in five dimensions admits an irreducible four-dimensional representation. The gamma matrices can be chosen as

$$\gamma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \tag{75}$$

where each entry is a two-by-two matrix and the σ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{76}$$

We shall need the following properties:

$$[\sigma_m, \sigma_n] \equiv \sigma_m\sigma_n - \sigma_n\sigma_m = 2i\varepsilon_{mnp}\sigma_p,$$

$$\{\sigma_m, \sigma_n\} \equiv \sigma_m\sigma_n + \sigma_n\sigma_m = 2\delta_{mn}, \tag{77}$$

as well as

$$\sigma_m\sigma_n = \frac{1}{2}(\{\sigma_m\sigma_n\} + [\sigma_m, \sigma_n]) = \delta_{mn} + i\epsilon_{mnp}\sigma_p. \tag{78}$$

If we use the Dirac matrices, Eq. (75), with Eq. (66) and the field

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{79}$$

we obtain

$$\begin{pmatrix} \sigma\cdot\mathbf{p} & \sqrt{2}p_5 \\ -\sqrt{2}p_4 & -\sigma\cdot\mathbf{p} \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{80}$$

or

$$\begin{aligned} \sigma\cdot\mathbf{p}\varphi + \sqrt{2}p_5\chi &= 0, \\ \sqrt{2}p_4\varphi + \sigma\cdot\mathbf{p}\chi &= 0. \end{aligned} \tag{81}$$

We will generalize these equations in the next section. By isolating χ in the first equation and substituting in the second, and using the embedding of Eq. (15), we obtain the anticipated result, $(2mE - \mathbf{p}^2)\varphi = 0$.

4. DIRAC EQUATION WITH AN EXTERNAL GAUGE FIELD

4.1. Abelian Gauge Field

First we investigate the interaction of the Dirac-like field with an external electromagnetic field. The latter comes into play by generalizing the Lagrangian in Eq. (67) to

$$\mathcal{L} = \bar{\Psi}(i\not{D})\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{82}$$

where the covariant derivative is

$$D_\mu = \partial_\mu - iqA_\mu, \quad \mu = 1, \ldots, 5 \tag{83}$$

so that we just have to perform the following minimal substitution into (66):

$$p_\mu \to \pi_\mu \equiv -iD_\mu = p_\mu - qA_\mu. \tag{84}$$

The constant q can be related to the elementary charge e and will be defined below, and the five-potential A_μ is the gauge field. As usual we have

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{85}$$

Then Eq. (81) is replaced by

$$\sigma \cdot \pi\varphi + \sqrt{2}\pi_5\chi = 0,$$
$$\sqrt{2}\pi_4\varphi + \sigma \cdot \pi\chi = 0 \tag{86}$$

These two equations can be rearranged as

$$\left[\pi_4 - \frac{1}{4}\sigma \cdot \pi(\pi_5)^{-1}\sigma \cdot \pi\right]\varphi = 0. \tag{87}$$

Let us consider the embedding defined in Eq. (15).

$$\pi_\mu = (\mathbf{p} - q\mathbf{A}, -E - qA_4, -m - qA_5), \tag{88}$$

with Eq. (87) to give

$$E = \frac{1}{2m}\sigma \cdot \pi\left(1 + \frac{qA_5}{m}\right)^{-1}\sigma \cdot \pi - qA_4. \tag{89}$$

Using the weak field approximation, $qA_5 \ll m$, we may use the expansion

$$\left(1 + \frac{qA_5}{m}\right)^{-1} \approx 1 - \frac{q}{m}A_5 + \frac{q^2}{m^2}A_5^2 - \frac{q^3}{m^3}A_5^3 + - \cdots, \tag{90}$$

so that Eq. (89) becomes

$$E = \frac{1}{2m}(\sigma \cdot \pi)^2 - qA_4 - \frac{q}{2m^2}\sigma \cdot \pi(A_5)\sigma \cdot \pi + \frac{q^2}{2m^3}\sigma \cdot \pi(A_5)^2\sigma \cdot \pi - + \cdots$$
$$+ (-1)^K\frac{q^K}{2m^{K+1}}\sigma \cdot \pi(A_5)^K\sigma \cdot \pi + \cdots \tag{91}$$

The leading term is

$$E^{(1)} = \frac{1}{2m}(\sigma \cdot \pi)^2 - qA_4. \tag{92}$$

(Let us just remark that it corresponds to the embedding of Eq. (49), which describes the magnetic limit.) From Eq. (78) we find

$$(\sigma \cdot \pi)^2 = \pi^2 + i\sigma \cdot (\pi \times \pi) = \pi^2 - q\sigma \cdot \mathbf{B}, \tag{93}$$

where we have used the definition of magnetic field

$$\mathbf{B} \equiv i\mathbf{p} \times \mathbf{A}. \tag{94}$$

Then Eq. (92) reduces to

$$E^{(1)} = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 - \frac{q}{2m}\sigma \cdot \mathbf{B} - qA_4, \tag{95}$$

and by defining

$$
\begin{aligned}
E &\equiv i\partial_t, \\
q &\equiv e/c, \\
A_4 &\equiv -c\Phi,
\end{aligned}
\tag{96}
$$

Eq. (95) becomes the well-known *Pauli equation*

$$
i\partial_t\varphi = \left[\frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 - \frac{e}{2mc}\sigma\cdot\mathbf{B} + e\Phi\right]\varphi. \tag{97}
$$

As remarked by Lévy-Leblond, the second term of this Hamiltonian shows the existence of an intrinsic magnetic moment equal to $\frac{e}{2mc}\sigma$ for the electron so that its gyromagnetic ration is $\frac{e}{mc}$, with a Landé factor $g_s = 2$, thereby proving it to be a *nonrelativistic* property of the electron (Lévy-Leblond, 1967).

Now let us turn to the next-to-leading order, that is, the first three terms of Eq. (91) with the vector field $\mathbf{A}$ set equal to zero to isolate the correction terms because of a weak electrostatic potential Φ.

$$
E^{(2)} = \frac{1}{2m}(\sigma\cdot\mathbf{p})^2 - qA_4 - \frac{q}{2m^2}\sigma\cdot\mathbf{p}A_5\sigma\cdot\mathbf{p}. \tag{98}
$$

By using Eq. (78) we find

$$
\begin{aligned}
\sigma\cdot\mathbf{p}A_5\sigma\cdot\mathbf{p} &= \sigma_m p_m A_5 \sigma_n p_n, \\
&= \sigma_m\sigma_n p_m A_5 p_n, \\
&= (\delta_{mn} + i\epsilon_{mnp}\sigma_p)p_m A_5 p_n, \\
&= p_m A_5 p_m + i\epsilon_{mnp}\sigma_p p_m A_5 p_n, \\
&= A_5\mathbf{p}^2 + (\mathbf{p}A_5)\cdot\mathbf{p} + i\sigma\cdot[(\mathbf{p}A_5)\times\mathbf{p}],
\end{aligned}
\tag{99}
$$

(hereafter we use summation over repeated indices, unless specified otherwise) so that

$$
E^{(2)} = \frac{\mathbf{p}^2}{2m} - qA_4 - \frac{q}{2m^2}A_5\mathbf{p}^2 - \frac{q}{2m^2}(\mathbf{p}A_5)\cdot\mathbf{p} - i\frac{q}{2m^2}\sigma\cdot[(\mathbf{p}A_5)\times\mathbf{p}]. \tag{100}
$$

(We recall that each term in Eq. (99) is an operator acting on a function.)

Now, in order to make further progress, let us consider A_5 as a spherically symmetric scalar potential

$$
A_5 = -\frac{1}{c}\Phi(r), \tag{101}
$$

so that the last term of Eq. (99) describes the spin–orbit interaction. Indeed

$$
\begin{aligned}
i\sigma \cdot [(\mathbf{p}A_5) \times \mathbf{p}] &= -\frac{i}{c}\sigma \cdot [(\mathbf{p}\Phi) \times \mathbf{p}], \\
&= -\frac{1}{c}\sigma \cdot [(\nabla\Phi) \times \mathbf{p}], \quad \mathbf{p} = -i\nabla, \\
&= -\frac{1}{c}\frac{d\Phi}{dr}\sigma \cdot \left(\frac{\mathbf{x}}{r} \times \mathbf{p}\right), \quad \nabla\Phi = \frac{d\Phi}{dr}\frac{\mathbf{x}}{r}, \\
&= -\frac{1}{c}\frac{1}{r}\frac{d\Phi}{dr}\sigma \cdot \mathbf{L}, \quad \mathbf{L} = \mathbf{x} \times \mathbf{p},
\end{aligned} \tag{102}
$$

so that by using Eqs. (96) and (101) the last term of Eq. (100) is the *spin–orbit interaction* term and we obtain

$$
E^{(2)} = \frac{\mathbf{p}^2}{2m} + e\Phi + \frac{e}{m^2c^2}\frac{1}{r}\frac{d\Phi}{dr}\mathbf{S} \cdot \mathbf{L} + \frac{e}{2m^2c^2}(\Phi\mathbf{p}^2 + (\mathbf{p}\Phi) \cdot \mathbf{p}), \tag{103}
$$

where $\mathbf{S} \equiv \frac{1}{2}\sigma$. Note that the factor 2 is missing; it is probable that this factor is explained by a purely relativistic effect, the Thomas precession, whereas we currently use a nonrelativistic setting. This feature is most desirable. Equation (103) contains more terms than Lévy-Leblond's, who claimed, for instance, that the spin–orbit interaction could not be described in a fully Galilean context (Lévy-Leblond 1967). However, as mentioned in the conclusion, other authors have obtained even more terms than we did here.

There are two other terms traditionaly found from the nonrelativistic limit of the Dirac equation: the *Darwin term* and the mass–velocity term. The mass–velocity term $(-\mathbf{p}^4/8m^3c^2)$ does not follow from our equations, which are quadratic in momentum. The third and fourth terms of Eq. (100) come from $p_m A_5 p_m$ (sum over indices). Therefore the last two terms of Eq. (103) are

$$
(\Phi\mathbf{p}^2 + (\mathbf{p}\Phi) \cdot \mathbf{p})\varphi = \mathbf{p} \cdot (\Phi\mathbf{p}\varphi). \tag{104}
$$

Therefore we express Eq. (103) as

$$
E^{(2)}\varphi = -\frac{\nabla^2\varphi}{2m} + e\Phi\varphi + \frac{e}{m^2c^2}\frac{1}{r}\frac{d\Phi}{dr}\mathbf{S} \cdot \mathbf{L}_\varphi - \frac{e}{2m^2c^2}\nabla \cdot (\Phi\nabla\varphi). \tag{105}
$$

Let us notice the contrast between the Hamiltonian above and Lévy-Leblond's results (1967) according to which neither the spin–orbit coupling nor the Darwin term can be obtained from a purely nonrelativistic Galilean theory. Here we could obtain the spin–orbit interaction. However, Nikitin and Fushchich (1980) went even further and have included the dipole, quadrupole, spin–orbit, and Darwin couplings of the particle to an external electromagnetic field.

4.2. Non-Abelian Gauge Field

Let us conclude by generalizing the previous discussion to the interaction with a *non-Abelian* gauge field. Let us denote the gauge group by G, generated by elements $\{t^a\}$ having commutation relations

$$[t^a, t^b] = i f_{abc} t^c, \quad a, b, c, = 1, \cdots, \dim G. \tag{106}$$

The Lagrangian (82) is replaced by

$$\mathcal{L} = \bar{\Psi}(i \not{D})\Psi - \frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu}, \tag{107}$$

where the covariant derivative takes the form

$$D_\mu = \partial_\mu - i g A_{a\mu} t^a. \tag{108}$$

This implies that one must replace Eq. (84) with

$$\pi_\mu = p_\mu - g A_{a\mu} t^a. \tag{109}$$

The field strength tensor (85) is generalized as usual:

$$F_{a\mu\nu} = \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g f_{abc} A_{b\mu} A_{c\nu}. \tag{110}$$

Substituting $p \to \pi$ into Eq. (66) and using again Eq. (75), we obtain equations similar to Eqs. (86). Next we define the embedding as in Eq. (15) and find a generalization of Eq. (88)

$$\pi_\mu = (\mathbf{p} - g \mathbf{A}_a t^a, -E - g A_{a4} t^a, -m - g A_{a5} t^a). \tag{111}$$

Then Eq. (87) leads to the operator equation

$$E + g A_{4a} t^a - \frac{1}{2m} \sigma \cdot \pi \left(1 + \frac{g}{m} A_{5a} t^a\right)^{-1} \sigma \cdot \pi = 0. \tag{112}$$

Next we use an expansion similar to Eq. (90)

$$\left(1 + \frac{g A_{5a} t^a}{m}\right)^{-1} \approx 1 - \frac{q}{m} A_{5a} t^a + \left(\frac{g}{m}\right)^2 (A_{5a} t^a)^2 - \left(\frac{g}{m}\right)^3 (A_{5a} t^a)^3 + - \cdots, \tag{113}$$

so that

$$\begin{aligned} E = {} & \frac{1}{2m} (\sigma \cdot \pi)^2 - q A_{4a} t^a - \frac{g}{2m^2} \sigma \cdot \pi A_{5a} t^a \sigma \cdot \pi + \frac{g^2}{2m^3} \sigma \cdot \pi (A_{5a} t^a)^2 \sigma \cdot \pi + \\ & - \frac{g^3}{2m^4} \sigma \cdot \pi (A_{5a} t^a)^3 \sigma \cdot \pi + - \cdots + (-1)^K \frac{g^K}{2m^{K+1}} \sigma \cdot \pi (A_{5a} t^a)^K \sigma \cdot \pi + \cdots \end{aligned} \tag{114}$$

As in the Abelian case, the next step is to analyze each term of this expansion. The first term is

$$\begin{aligned}(\sigma\cdot\pi)^2 &= [\sigma\cdot(\mathbf{p}-g\mathbf{A}_a t^a)]^2,\\ &= (\mathbf{p}-g\mathbf{A}_a t^a)^2 - ig\sigma\cdot(\mathbf{p}\times\mathbf{A}_a+\mathbf{A}_a\times\mathbf{p})t^a + ig^2\sigma\cdot(\mathbf{A}_a\times\mathbf{A}_b)t^a t^b.\end{aligned} \tag{115}$$

Calculations similar to the Abelian case show that

$$(\sigma\cdot\pi)^2 = (\mathbf{p}-g\mathbf{A}_a t^a)^2 - g\sigma\cdot\mathbf{B}_a t^a + \frac{i}{2}g^2\sigma\cdot(\mathbf{A}_a\times\mathbf{A}_b)([t^a,t^b]+\{t^a,t^b\}), \tag{116}$$

where, in analogy with Eq. (94), we use the notation

$$\mathbf{B}_a \equiv i\mathbf{P}\times\mathbf{A}_a. \tag{117}$$

Then we obtain a *non-Abelian* version of the Pauli equation (97)

$$\begin{aligned}E^{(1)} &= \frac{1}{2m}(\mathbf{p}-g\mathbf{A}_a t^a)^2 - gA_{4a}t^a - \frac{g}{2m}\sigma\cdot\mathbf{B}_a t^a\\ &\quad + \frac{i}{4m}g^2\sigma\cdot(\mathbf{A}_a\times\mathbf{A}_b)([t^a,t^b]+\{t^a,t^b\}).\end{aligned} \tag{118}$$

From the next term in Eq. (98), we consider the next term of Eq. (114) with $\mathbf{A}=0$,

$$E^{(2)} = -gA_{4a}t^a + \frac{1}{2m}(\sigma\cdot\mathbf{p})^2 - \frac{g}{2m^2}\sigma\cdot\mathbf{p}A_{5a}\sigma\cdot\mathbf{p}t^a. \tag{119}$$

Using a treatment entirely similar to the one leading to Eq. (99) we find

$$\sigma\cdot\mathbf{p}A_{5a}\sigma\cdot\mathbf{p}t^a = A_{5a}\mathbf{p}^2 t^a + (\mathbf{p}A_{5a})\cdot\mathbf{p}t^a + i\sigma\cdot[(\mathbf{p}A_{5a})\times\mathbf{p}t^a]. \tag{120}$$

Whereas the last term would lead to a non-Abelian version of the spin–orbit interaction, the first two terms would describe some effect reminiscent of the Darwin interaction.

For example, with the group SU(2) we have $t^a \equiv \frac{1}{2}\sigma_a$ where σ is a Pauli matrix with $[t^a,t^b] = i\varepsilon_{abc}t^c$ and $\{t^a,t^b\} = \frac{1}{2}\delta_{ab}$. Then Eq. (118) becomes

$$E^{(1)} = -gA_{4a}t^a + \frac{1}{2m}(\mathbf{p}-g\mathbf{A}_a t^a)^2 - \frac{g}{2m}\sigma\cdot\mathbf{B}_a t^a - \frac{g^2}{4m}\varepsilon_{abc}\sigma\cdot(\mathbf{A}_a\times\mathbf{A}_b)t^c. \tag{121}$$

Note that the last term does not appear in Eqs. (95) or (97).

5. CONCLUDING REMARKS

First we repeat that the original aspect of this paper lies in the covariant approach to Galilei-invariant equations, rather than in original equations describing some new physics. The purpose is to have both Lorentz and Galilei covariance formulations as similar as possible, in order to provide a guiding principle to write down dynamical equations of nonrelativistic phenomena as well as relativistic ones. In fact both kinematical theories can be described by starting within a five-dimensional Riemannian manifold with metric, Eq. (3). We have derived many results that have been obtained previously from a noncovariant approach: Maxwell equations, Dirac equation for free fermions and for fermions coupled to an external, Abelian and non-Abelian, gauge field. Our physically original result is the nonrelativistic wave equations with coupling to a non-Abelian external gauge field.

Let us conclude by recalling some differences between the Hamiltonians obtained in section 4 and similar equations contained in the literature. All agree on the Pauli equation, and the associated appearance of the correct Landé factor. Lévy-Leblond (1967) has found that neither the spin–orbit coupling nor the Darwin term can be obtained from a purely nonrelativistic Galilean theory for an elementary particle, whereas we have obtained the spin–orbit interaction in a theory with Galilean covariance. On the other hand Fushchich and Nikitin have obtained Hamiltonians allowing for the dipole, quadrupole, spin–orbit, and Darwin couplings of particles to an external electromagnetic field to exist (Nikitin and Fuschich, 1980; Fuschich and Nikitin, 1994). Recently we have learned about a preprint in which a study similar to ours is performed and in which solutions involve the existence of negative energy states; these would be interpreted as describing antiparticles (Horzela and Kapuścik, 2002). In addition to clarifying these aspects, further work can be done in interpreting Hamiltonians such as those found in Section 5, based on non-Abelian groups. Interpretation of the various terms of the SU(2) Hamiltonian still needs to be found. Study of larger groups, such as SU(3), can be carried out in a similar way.

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